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The stability of three-dimensional time-periodic flows with ellipsoidal stream surfaces

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Most steady flows with constant vorticity and elliptical streamlines are known to be unstable. These, and certain axisymmetric time-periodic flows, can be analysed by Floquet theory. However, Floquet theory is inapplicable to other time-periodic flows that yield disturbance equations containing a quasi-periodic, rather than periodic, function. A practical method for surmounting this difficulty was recently given by Bayly, Holm & Lifschitz. Employing their method, we determine the stability of a class of three-dimensional time-periodic flows: namely, those unbounded flows with fixed ellipsoidal stream surfaces and spatially uniform but time-periodic strain rates. Corresponding, but bounded, flows are those within a fixed ellipsoid with three different principal axes. This is perhaps the first exact stability analysis of non-reducibly three-dimensional and time-dependent flows. Though the model has some artificial features, the results are likely to shed light on more complex systems of practical interest.

1. Introduction

The so-called 'elliptical instability', first revealed computationally and analytically by Pierrehumbert (1986) and Bayly (1986) respectively, has engendered much subsequent interest. This instability, most easily analysed for unbounded steady primary flows with self-similar elliptical streamlines, is essentially inviscid in character and normally exhibits a broad band of unstable wavenumbers with surprisingly large growth rates. The instability mechanism is most simply viewed as a resonance, in which disturbances of plane-wave form are advected by the primary flow and so experience periodic straining on account of the ellipticity. Suitably oriented wavenumbers have natural frequencies that are closely tuned to this periodic straining and so are unstable.

The presence of boundaries modifies this picture in two respects. Firstly, the supposed elliptical flow may not satisfy the appropriate viscous boundary conditions at the container walls; and, secondly, the assumed plane-wave form of disturbances is likely to be incompatible with both inviscid and viscous wall boundary conditions of any realistic sort. Nevertheless, there is evidence that the basic elliptical instability mechanism still operates importantly for bounded flows. For instance, Waleffe (1990) has analysed the stability of a rotating flow within a container of elliptical cross-section that is almost circular. He established that the growth rates of the admissible linear modes agree well with those given by the corresponding unbounded theory whenever the characteristic length scale of the disturbance is small compared with the cylinder radius. Additional support for this is provided by the ingenious experiments of Malkus (1989), conducted using a cylindrical tank with deformable walls.

It has been conjectured that the elliptical instability and its near-relatives may have an important part to play in the continuous reinforcement of three-dimensionality within turbulent flows, over a quite broad band of length scales, since there are certainly local regions within such flows that resemble the elliptical configuration. It has also been suggested that the elliptical Kelvin 'cat's eyes' configuration associated with two-dimensional Tollmien–Schlichting waves in unstable boundary layers provides the source, via the elliptical instability, of the observed rapid growth of three-dimensional disturbances (though the present writers find other, longer-established, explanations more plausible in the latter case). Such speculations have in turn motivated theoretical studies of other simple models: for instance, Craik (1989), Mansour & Lundgren (1990), Miyazaki & Fukumoto (1992), Craik & Allen (1992). The last of these addressed the stability of an otherwise uniformly rotating flow subjected to periodic straining: there, the resonance is not primarily associated with streamline ellipticity, but rather with the externally imposed oscillation. By such studies, some insight may be gained into the likely robustness of the elliptical and related instabilities to external factors such as time-dependent straining and tilting of the flow. Clearly, in view of the complexity of turbulent flows, any effective mechanism must remain robust under such external influences.

A perhaps greater motivation for the present work was the desire to provide definitive solutions, preferably without approximations, to simple but clear-cut problems of a kind little studied previously, and that require novel methods of analysis. A particular analytical difficulty was encountered, but not resolved, by Craik & Allen (1992): though they could successfully analyse a class of axisymmetric time-oscillatory primary flows by means of Floquet theory, corresponding non-axisymmetric problems could not be treated similarly and they were forced to make approximations in such cases. Since then, a method has been developed by Bayly, Holm & Lifschitz (1996), to examine certain two-dimensional flows, that applies equally well in three dimensions. We use this here to give what we believe to be the first full stability analysis of an irreducibly three-dimensional and time-periodic flow.

The class of flows that we examine in detail is fully described later. Briefly, it is the class of unbounded flows equivalent to those within a fixed ellipsoidal container with three different principal axes, and having spatially uniform rates of strain at each instant. Such flows are not normally steady, but precess periodically in time. Though these flows admit the elliptical instability, they do so within a time-periodic environment where both the instantaneous axis of rotation and the magnitude of the vorticity must change. The influence of temporal evolution on the stability characteristics of the flow is thereby explored.

The stability of steady, unbounded, incompressible flows with spatially-uniform strain rates has been studied by Kelvin (1887), Lagnado, Phan-Thien & Leal (1984), Craik & Criminale (1986), Bayly (1986), Craik (1989), Waleffe (1990), Miyazaki & Fukumoto (1992) and others. In absence of body forces, all such primary flows, whether steady or not, necessarily have the form (see e.g. Craik & Criminale 1986 or Craik & Allen 1992):

$$\boldsymbol{U}(\boldsymbol{x},t) = \boldsymbol{S}(t)\boldsymbol{x} + \boldsymbol{U}^0(t) \tag{1.1}$$

where $\boldsymbol{S}(t)$ is a 3×3 matrix with zero trace and such that

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} + \boldsymbol{S}^2 = \boldsymbol{M}(t) \quad \text{(symmetric)} \tag{1.2}$$

and $\boldsymbol{M}(t)$ is an arbitrary symmetic matrix. Inclusion of body forces modifies the latter equation, but with no ensuing additional difficulties: see Craik (1989), Miyazaki & Fukumoto (1992).

Three-dimensional disturbances are taken in the 'plane-wave' form

$$\boldsymbol{u}'(\boldsymbol{x},t) = \mathrm{Re}\{\boldsymbol{v}(t)\exp[\mathrm{i}\boldsymbol{\alpha}(t)\cdot\boldsymbol{x}]\} \tag{1.3}$$

for each given $\boldsymbol{v}(0)$, $\boldsymbol{\alpha}(0)$. The wavenumber advection equation is then

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} = -\boldsymbol{S}^{tr}\boldsymbol{\alpha} \tag{1.4}$$

where the superscript '*tr*' denotes transpose. This equation represents advection of the disturbance by the primary flow in such a way that material planes remain plane at all later times, though with differing orientation.

When a *steady* primary flow $\boldsymbol{U}(\boldsymbol{x})$ has closed elliptical streamlines, the wavenumber vector $\boldsymbol{\alpha}(t)$ is found exactly from (1.4), as an explicit periodic function: cf. Craik & Criminale (1986), Bayly (1986).

The influence of kinematic viscosity ν on $\boldsymbol{v}(t)$ is easily eliminated from the analysis by incorporating a multiplicative damping factor,

$$\boldsymbol{v}(t) = \exp\left\{-\nu\int\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\mathrm{d}t\right\}\boldsymbol{v}_{inv}(t).$$

Then, the amplitude components $\boldsymbol{v}_{inv}(t)$ of the reduced inviscid problem are found from the solution of an associated Floquet problem

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \boldsymbol{T}(t)\boldsymbol{B}, \qquad \boldsymbol{B}(0) = \boldsymbol{I}_2, \tag{1.5}$$

where $\boldsymbol{I}_2$ is the unit matrix and $\boldsymbol{T}(t)$ is a 2×2 time-periodic matrix known in terms of $\boldsymbol{\alpha}(t)$. The reduction to a two-dimensional system and the precise form of the matrix $\boldsymbol{T}$(t) are described later, in §4.

To calculate the Floquet exponents, and so determine whether the chosen initial disturbance is stable or unstable, it is necessary only to compute over one time period (say $0 < t < \tau$) to calculate the trace of $\boldsymbol{B}$ at $t = \tau$: see Bayly (1986), Craik (1989), Waleffe (1990). Alternatively, the problem may be cast as a single Schrödinger-type equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + V(t)q = 0 \tag{1.6}$$

where the potential $V(t)$ is τ-periodic. Typically, there are bands of instability corresponding to various internal resonances, much as for the Mathieu equation. Instability is suppressed only when these resonance criteria cannot be met for the flow in question.

It is a remarkable fact that the above theoretical description is *exact*, provided that only one 'plane-wave' disturbance is present, and however large its amplitude. It remains exact for superpositions of plane waves with identical wavenumber orientations, i.e. for general 'planar' disturbances. But other combinations of 'plane-wave' Fourier modes, with wavenumbers not all parallel to each other, give rise to non-linear interactions; consequently, for such general disturbances, the theory must be interpreted as a linearized approximation valid for sufficiently small amplitudes.

A class of unbounded three-dimensional time-periodic primary flows was considered by Craik & Allen (1992): see also Mansour & Lundgren (1990) and Craik (1995). This class comprises flows with spatially uniform vorticity distorted by a pulsating stagnation-point flow. Then, the wavenumber vector of the disturbance is doubly periodic (broadly speaking, with one period associated with advection by the mean vorticity and the other with the imposed pulsations), and these separate periodicities are not necessarily rationally related. Then, the matrix $\boldsymbol{T}(t)$ and potential $V(t)$ are not necessarily t-periodic, but may instead be quasi-periodic containing separate periodic functions with incommensurate periods. Floquet theory can still be used when all the periods present are rationally related, but numerical integration must then be carried out over $0 < t < M$ where M is the lowest common multiple of the periods. If M is very large, this integration becomes impracticable; and when the periods are incommensurate (i.e. M is infinite) Floquet theory is unavailable.

It was found by Craik & Allen (1992) that a sub-class of time-periodic flows that are *axisymmetric* is still amenable to Floquet analysis because the matrix $\boldsymbol{T}(t)$ and potential $V(t)$ are again τ-periodic even though $\boldsymbol{\alpha}(t)$ is not. They also showed that, with weak but non-axisymmetric periodic pulsations, an approximate procedure yields the strongest of the instability bands. Other apparently time-periodic flows, within a rotating and precessing axisymmetric ellipsoid, are also reducible to Floquet analysis because a rotating reference frame may be chosen within which these flows are steady, at the expense only of introducing a Coriolis force: see Kerswell (1993).

2. Quasi-periodic potentials

The question of how to cope with quasi-periodic forcing is an important one, for its resolution allows the stability of many more flows to be determined. This has recently been addressed by Bayly *et al.* (1996). Informed by theoretical studies of the Schrödinger equation with quasi-periodic potentials by Johnson & Moser (1982) and Simon (1982), they devised a convenient and convincing computational method. With (a, b) defined as $(q, dq/dt)$, the Schrödinger equation (1.6) is first recast as

$$\begin{pmatrix} \mathrm{d}a/\mathrm{d}t \\ \mathrm{d}b/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -V & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then, polar variables $c(t)$, $s(t)$ are introduced such that

$$a(t) = \exp[s(t)]\sin[c(t)], \qquad b(t) = \exp[s(t)]\cos[c(t)].$$

These satisfy

$$\left.\begin{aligned} 2\frac{\mathrm{d}c}{\mathrm{d}t} &= (1+V) + (1-V)\cos(2c), \\ 2\frac{\mathrm{d}s}{\mathrm{d}t} &= (1-V)\sin(2c). \end{aligned}\right\} \tag{2.1}$$

Note that $s(t)$ does not arise in the former. Clearly, the flow is unstable if, for some initial data, $s(t)$ grows without bound as t approaches infinity.

Now, it is known from the theory of quasi-periodic potentials that, with rather weak restrictions on $V(t)$, the limiting values

$$W = \lim_{t\to\infty}\left\{\frac{c(t)}{t}\right\}, \qquad I = \lim_{t\to\infty}\left\{\frac{s(t)}{t}\right\}$$

exist and are independent of the initial values $c(0)$, $s(0)$. The limits W and I are respectively known as the 'winding number' and 'growth rate'. For instability, it is sufficient that $I > 0$. Bayly *et al.* (1995) have convincingly shown that I can be approximated by computations over finite intervals, as convergence is fairly rapid.

Two families of two-dimensional flows with time-periodic strain rates were examined by Bayly *et al.* The first family, chosen for analytical convenience, contains steady elliptical flow as a limiting case but leads in general to a potential $V(t)$ that contains functions with two periods. Illustrative examples are given where the periods are rationally and irrationally related. A single parameter that characterizes the orientation of the initial three-dimensional wavenumber $\boldsymbol{\alpha}(0)$ is the angle θ between it and the normal to the plane of the basic flow. Estimates of W and I were found, for various θ in $[0, \frac{1}{2}\pi]$, by computing for times long enough that the limits are convincingly approached. Graphs show their estimates of W and I versus $\mu = \cos\theta$.

When the periods are rationally related, there are several distinct bands of instability, one of which is much stronger and wider than the rest. (The number of such bands, which can be predicted theoretically, is confirmed; but care is needed to discover the weakest of these, as they have very small growth rates.) Within each instability band, the winding number W takes a different constant value, and W increases monotonically with μ between the bands. Identical results may be found by Floquet theory.

When the periods are irrationally related (and Floquet theory is unavailable) estimates of W and I are plotted similarly, and several instability bands are again evident. However, it is now impossible to find all instability bands. Though these are infinite in number, and dense on intervals of the μ-axis, all but a few are too weak to be of physical significance, in view of the expected damping role of viscosity. Again, the winding number W is constant within each band of instability.

The second class of flows considered by Bayly *et al.* is the family of time-periodic uniform-vorticity cores associated with externally strained Kirchhoff–Kida vortices (Kida 1981). These elliptically-shaped vortices can change their shape as they rotate or librate. The 'geometrical optics' approximation is invoked, to justify treatment of these flows as spatially infinite when the scale of the disturbance wavelength is small compared with the minor axis of the vortex core. In a similar manner, Bayly *et al.* then demonstrate that almost all members of this family of inviscid flows are strongly unstable to three-dimensional plane-wave disturbances.

3. Three-dimensional time-periodic flows

Two other classes of time-dependent primary flows that are amenable to the above method are briefly discussed by Craik (1995), but without actual results. The first class is that also discussed by Craik & Allen (1992). These have time-periodic principal rates of strain $a, b, -a-b$ constantly directed along the coordinate axes, so that the matrix $\boldsymbol{S}$ of (1.1) is

$$\boldsymbol{S} = \begin{pmatrix} a(t) & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & b(t) & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & -a(t)-b(t) \end{pmatrix}, \qquad (3.1)$$

$$a(t) = a_0 \cos \Omega t, \qquad b(t) = b_0 \cos \Omega t, \qquad \Omega \quad \text{constant.}$$

Because of (1.2), the off-diagonal components $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$, which yield one half of the vorticity, are

$$\omega_1 = \omega_{10} \exp(a_0 P), \quad \omega_2 = \omega_{20} \exp(b_0 P), \quad \omega_3 = \omega_{30} \exp[-(a_0 + b_0)P], \qquad (3.2)$$

where $P(t) = \Omega^{-1} \sin \Omega t$ and the ω_{i0} are constants. Clearly, the periodic extension and contraction of vorticity is non-sinusoidal. Disturbances of the form (1.3) may now be introduced. The resulting evolution equation (1.4) for the three components of wavenumber $\boldsymbol{\alpha}(t)$ have quasi-periodic solutions, with analytical expressions available only in the axisymmetric case $a_0 = b_0$, $\omega_{10} = \omega_{20} = 0$ studied by Craik & Allen. The corresponding amplitude equations (1.5) or (1.6) in non-axisymmetric cases have a quasi-periodic matrix $\boldsymbol{T}(t)$ or potential $V(t)$ expressible in terms of the $\boldsymbol{\alpha}$-components. Computation of the winding number W and growth rate I can then proceed as described above. Though details are not yet available, almost all such flows are likely to be strongly unstable.

The second class of basic flows is that examined here. These are the three-dimensional unbounded time-periodic flows that correspond to (bounded) flows with spatially uniform vorticity within an ellipsoid, say $(x/a)^2+(y/b)^2+(z/c)^2 = \text{constant}$: that is to say, 'swirling' flows with fixed, similar, ellipsoidal stream surfaces. Stability of steady and uniformly precessing flows, within fixed, rotating or precessing ellipsoids, has already been examined by Gledzer & Ponomarev (1992) and Kerswell (1993); but intrinsically time-periodic basic flows have not been examined before. If the 'geometrical optics' approximation is invoked for small-scale disturbances, the above methods yield results valid for finite as well as for infinite domains.

All possible primary flows with spatially uniform strain rates in this geometry have velocity components $\boldsymbol{U} = \boldsymbol{S}(t)\boldsymbol{x}$ with

$$\boldsymbol{S} = \begin{pmatrix} 0 & \dfrac{a\omega_3(t)}{b} & -\dfrac{a\omega_2(t)}{c} \\ -\dfrac{b\omega_3(t)}{a} & 0 & \dfrac{b\omega_1(t)}{c} \\ \dfrac{c\omega_2(t)}{a} & -\dfrac{c\omega_1(t)}{b} & 0 \end{pmatrix} \tag{3.3}$$

where from (1.2)

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \left(\frac{c^2-b^2}{c^2+b^2}\right)\omega_2\omega_3, \quad \frac{\mathrm{d}\omega_2}{\mathrm{d}t} = \left(\frac{a^2-c^2}{a^2+c^2}\right)\omega_3\omega_1, \quad \frac{\mathrm{d}\omega_3}{\mathrm{d}t} = \left(\frac{b^2-a^2}{b^2+a^2}\right)\omega_1\omega_2. \tag{3.4}$$

For 'plane-wave' disturbances like (1.3), the wavenumber equation (1.4) becomes

$$\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t} = -\boldsymbol{\omega}\times\boldsymbol{\beta}, \quad \boldsymbol{\beta} \equiv [a\alpha_1, b\alpha_2, c\alpha_3], \quad \boldsymbol{\omega} \equiv [\omega_1, \omega_2, \omega_3]. \tag{3.5}$$

As described in the next section the corresponding amplitude equation (1.5) is found to reduce to the form

$$\begin{pmatrix} \mathrm{d}P/\mathrm{d}t \\ \mathrm{d}Q/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix}, \tag{3.6}$$

where $P \equiv \alpha_1 u_2 - \alpha_2 u_1$ and $Q \equiv |\boldsymbol{\alpha}(0)|u_2$ and the r_{ij} are known time-dependent functions. This is readily cast in a form similar to (2.1) with suitably defined modulus and phase variables. The winding number W and growth rate I may then be computed for chosen initial data and the stability of the flow established.

4. Formulation

It is known (Craik & Allen 1992) that the inviscid velocity components $\boldsymbol{v}_{inv} \equiv \boldsymbol{u}$ for the disturbance satisfy

$$\mathrm{d}\boldsymbol{u}/\mathrm{d}t + \Pi\boldsymbol{u} = 0, \qquad \Pi = \{\tau_{ij}\} \tag{4.1}$$

where

$$\tau_{ij} = \sigma_{ij} - \frac{2\alpha_i\alpha_k\sigma_{kj}}{(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})} \qquad \text{when} \qquad \mathbf{S} \equiv \{\sigma_{ij}\}.$$

By continuity we also have $\boldsymbol{u}\cdot\boldsymbol{\alpha} = 0$, i.e.

$$u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3 = 0, \tag{4.2}$$

which is automatically satisfied when (4.1) is. Seeking to reduce the three equations (4.1) to two coupled equations in just two unknowns, we introduce the dependent variables (cf. Waleffe 1990)

$$P \equiv \alpha_1 u_2 - \alpha_2 u_1, \qquad Q \equiv |\boldsymbol{\alpha}(0)| u_3. \tag{4.3}$$

From (1.4), (4.1) and (4.2), we obtain the two expressions

$$\dot{P} = -\frac{c\alpha_3 P}{(\alpha_1^2+\alpha_2^2)}\left(\frac{\alpha_1\omega_2}{a} - \frac{\alpha_2\omega_1}{b}\right) + \frac{Q}{|\boldsymbol{\alpha}(0)|}\left[-\alpha_3\omega_3\left(\frac{b}{a}+\frac{a}{b}\right)\right.$$
$$\left. -\alpha_2\omega_2\left(\frac{a}{c} - \frac{c\alpha_3^2}{a(\alpha_1^2+\alpha_2^2)}\right) - \alpha_1\omega_1\left(\frac{b}{c} - \frac{c\alpha_3^2}{b(\alpha_1^2+\alpha_2^2)}\right)\right], \tag{4.4}$$

and

$$\dot{Q} = \frac{|\boldsymbol{\alpha}(0)|\,P}{\alpha_1^2+\alpha_2^2}\left[\frac{c(\alpha_1^2+\alpha_2^2-\alpha_3^2)}{(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})}\left(\frac{\omega_1\alpha_1}{b} + \frac{\omega_2\alpha_2}{a}\right) + \frac{2\alpha_3\omega_3}{(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})}\left(\frac{\alpha_1^2 a}{b} + \frac{\alpha_2^2 b}{a}\right)\right]$$
$$+\frac{Q\alpha_3}{(\alpha_1^2+\alpha_2^2)(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})}\left[c(\alpha_1^2+\alpha_2^2-\alpha_3^2)\left(\frac{\alpha_1\omega_2}{a} - \frac{\alpha_2\omega_1}{b}\right)\right.$$
$$\left.+\frac{2(\alpha_1^2+\alpha_2^2)}{c}(-\alpha_1 a\omega_2 + \alpha_2 b\omega_1) + 2\alpha_1\alpha_2\alpha_3\omega_3\left(\frac{b}{a} - \frac{a}{b}\right)\right], \tag{4.5}$$

where the overdot ($\dot{\ }$) denotes $\mathrm{d}/\mathrm{d}t$. Here, the velocity components u_j have been eliminated. These two expressions are now in the form (3.6), with functions $r_{ij}(t)$ known in terms of the $\omega_j(t)$ and $\alpha_j(t)$.

Next, polar variables ρ, ϕ are introduced such that

$$P + \mathrm{i}Q = \rho \mathrm{e}^{\mathrm{i}\phi}. \tag{4.6}$$

This gives

$$\dot{\rho} = \cos\phi \dot{P} + \sin\phi \dot{Q}$$

from which we obtain

$$\dot{\rho} = \rho\left[r_{11}\cos^2\phi + r_{22}\sin^2\phi + (r_{12}+r_{21})\sin\phi\cos\phi\right]. \tag{4.7}$$

On setting $\rho = \rho(0)\mathrm{e}^{s(t)}$, equation (4.7) yields

$$\dot{s} = \left(\frac{r_{11}-r_{22}}{2}\right)\cos 2\phi + \left(\frac{r_{12}+r_{21}}{2}\right)\sin 2\phi + \left(\frac{r_{11}+r_{22}}{2}\right) \tag{4.8}$$

upon use of trigonometric identities. To obtain an expression for $\dot{\phi}$, first note that, from (4.6),

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}(\rho\cos\phi)}{\mathrm{d}t} = \dot{\rho}\cos\phi - \rho\sin\phi\dot{\phi} = (\dot{s}\cos\phi - \dot{\phi}\sin\phi)\rho.$$

Substitution for $\mathrm{d}P/\mathrm{d}t$ from (3.6) gives

$$\dot{\phi}\sin\phi = \dot{s}\cos\phi - r_{11}\cos\phi - r_{12}\sin\phi,$$

which leads to

$$\dot{\phi} = -r_{12} + \cot\phi\left[\left(\frac{r_{11}-r_{22}}{2}\right)(\cos 2\phi - 1) + \left(\frac{r_{12}+r_{21}}{2}\right)\sin 2\phi\right],$$

on using (4.8). We therefore have the two equations

$$\dot{\phi} = \left(\frac{r_{12}+r_{21}}{2}\right)\cos 2\phi - \left(\frac{r_{11}-r_{22}}{2}\right)\sin 2\phi + \left(\frac{r_{21}-r_{12}}{2}\right), \tag{4.9}$$

$$\dot{s} = \left(\frac{r_{12}+r_{21}}{2}\right)\sin 2\phi + \left(\frac{r_{11}-r_{22}}{2}\right)\cos 2\phi + \left(\frac{r_{11}+r_{22}}{2}\right). \tag{4.10}$$

From the theory of quasi-periodic potentials, the limits

$$W = \lim_{t\to\infty}\left(\frac{\phi(t)}{t}\right) \tag{4.11}$$

and

$$I = \lim_{t\to\infty}\left(\frac{s(t)}{t}\right) \tag{4.12}$$

are known to exist under fairly general conditions, including those here. It is these limits that we shall compute to establish the stability properties. A rigorous proof of the theorem establishing existence of these limits is analytically challenging (see Johnson & Moser 1982; Simon 1982) and readers wishing to explore this territory may wish to enlist the help of an experienced guide. In practice, our computations of $\phi(t)/t$ and $s(t)/t$ settled down to virtually constant values within fairly short times and there is no doubt that the limits exist in our problem.

In fact, unlike equation (7) of Bayly *et al.* the system (4.9) and (4.10) does not directly transform into the linear Schrödinger equation (1.6) for which the theory of quasi-periodic potentials was developed. Nevertheless the numerical results reported below conform with expectations. In particular, we find that the winding number W is constant in unstable regions and increases uniformly elsewhere as the control parameter varies. It is easy to see why this is so: for equations (4.4) and (4.5) can be reduced to Schrödinger form as follows. Elimination of Q yields a second-order equation for $P(t)$ in the form

$$\ddot{P} + p(t)\dot{P} + q(t)P = 0.$$

The substitution

$$P(t) = S(t)\exp\left[-\frac{1}{2}\int^{t} p(\tau)\mathrm{d}\tau\right]$$

gives the Schrödinger form

$$\ddot{S} + J(t)S = 0$$

where

$$J(t) \equiv q - \tfrac{1}{2}\dot{p} - \tfrac{1}{4}p^2.$$

Since $p(t)$ and $q(t)$ are quasi-periodic functions, $J(t)$ also acts as a quasi-periodic potential.

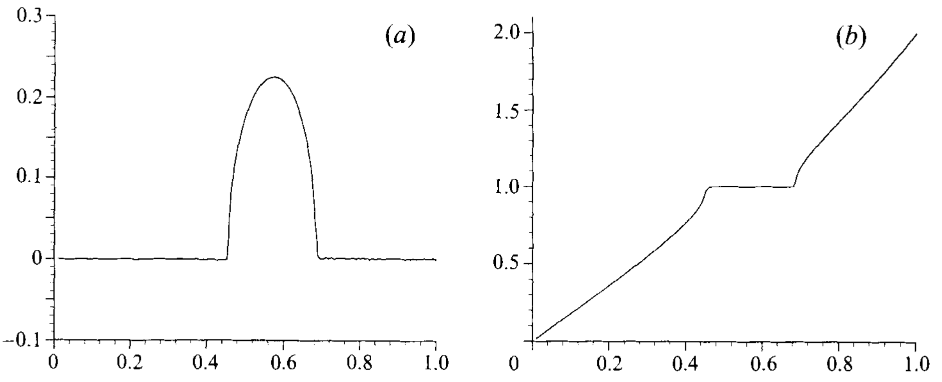

FIGURE 1. Growth rate I and winding number W versus $\cos\theta$ for $\omega_1(0) = 0$ and $\omega_3(0) = 1$, with $(a, b, c) = (3, 2, 1)$.

5. Numerical results

In order to find the limits W and I, the sets of equations (3.4) for $\boldsymbol{\omega}$, (1.4) for $\boldsymbol{\alpha}$ then (4.9) and (4.10), are integrated numerically over times sufficiently large that (4.11) and (4.12) are approached convincingly. We can take $|\boldsymbol{\omega}(0)| = 1$, $|\boldsymbol{\alpha}(0)| = 1$ and one of $\alpha_j(0) = 0$ without loss of generality. By choosing $\alpha_2(0)$ to be equal to zero, we can introduce the variable θ such that $\alpha_1(0) = \sin\theta$ and $\alpha_3(0) = \cos\theta$: this represents the angle between the initial wavevector of the disturbance and the z-coordinate axis. The initial values $s(0)$ and $\phi(0)$ can also be taken as zero without loss as the limits are independent of these values.

Upon declaration of a, b, c (which define the shape of the ellipsoid) and the initial values $\omega_{1,2}(0), \theta$, a numerical integration is initiated, generally over a time $t = 1000$, which was found to be sufficiently large for satisfactory convergence. The integration comprised a Merson form of the explicit Runga–Kutta method and consisted of a sequence of steps whose sizes were adapted throughout the integration in order that the final result should lie within a tolerance of 10^{-10}. Values of θ were chosen between 0 and $\frac{1}{2}\pi$ at intervals of 0.005 until the time limit was reached, and graphs were then plotted of $\cos\theta$ against the large-time limits, I and W, approached by $s(t)/t$ and $\phi(t)/t$ respectively.

Cases examined in detail have $(a, b, c) = (3, 2, 1)$ or (1,2,3), which are of course physically identical: others may be done similarly. Throughout we choose $\omega_2(0) = 0$, which entails some loss of generality; but other choices yield similar results. The evolution of the unstable regions is shown as the initial values of ω_1 and ω_3 are altered.

Figure 1 shows the stability properties when $(a, b, c) = (3, 2, 1)$ and $\omega_1(0) = 0$ and $\omega_3(0) = 1$. This corresponds to a steady, purely elliptical fluid flow centred around the z-axis, with no fluid rotation around the x- or y-axes. This is equivalent to the Floquet case examined by Bayly (1986) and, in the special case of no body force, by Craik (1989). The figure clearly shows that just one instability band exists for the region $0.45 < \theta < 0.68$: this is in agreement with the results of Bayly and of Craik.

We then examined what happens when the primary motion of the fluid is not purely in the (x, y)-planes. To do this, a small initial component of rotation $\omega_1(0)$ was imposed. This has the result of introducing a time-periodicity into the basic flow which causes a marked change in the stability of the system. However, a numerical difficulty was encountered in this and similar cases, resulting in breakdown of the

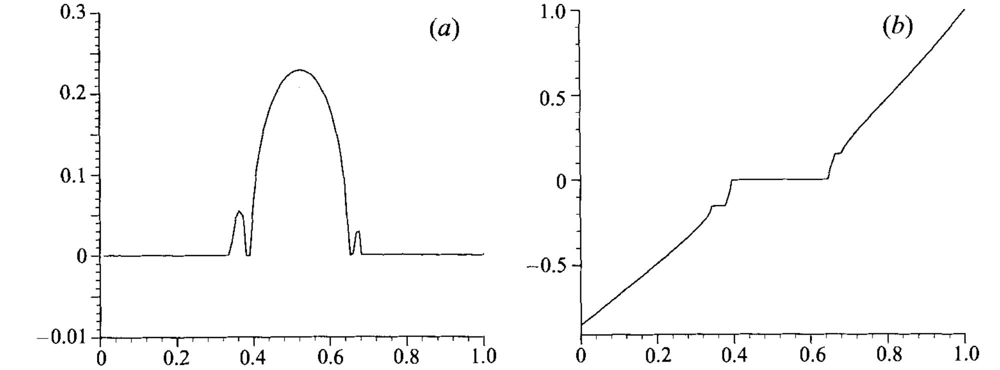

FIGURE 2. Growth rate I and winding number W versus $\sin\theta$ for $\omega_1(0) = 0.9972$ and $\omega_3(0) = 0.075$, with $(a, b, c) = (1, 2, 3)$.

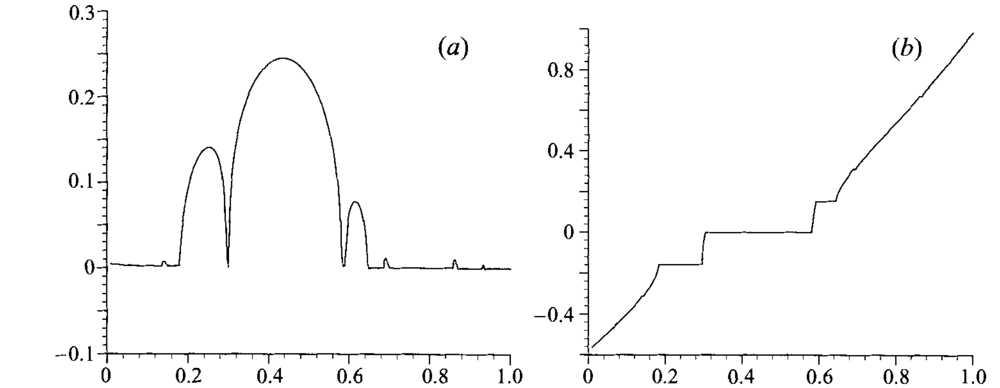

FIGURE 3. Same as figure 2 but for $\omega_1(0) = 0.9798$ and $\omega_3(0) = 0.20$.

calculation of the winding number W for small θ. The source of the difficulty seems worth recording. This corresponds to cases where the initial disturbance wavevector lies almost along the z-axis. For these, difficulties arise since α_1 and α_2 can pass through, or very close to, zero at the same time. This causes problems with the calculation of the r_{ij} expressions since they contain divisors in $\alpha_1^2 + \alpha_2^2$. The problem was easily overcome by switching the greatest and least axes of the ellipsoid (3 and 1) to lie along the z- and x-axes respectively (instead of the x- and z-axes). The same numerical integration was then performed, but due to the switch, the initial values of ω_{10} and ω_{30} were also interchanged to correspond to the same physical state as before. Correspondingly, the growth rates and winding numbers are plotted against $\sin\theta$ instead of $\cos\theta$ in order that the growth-rate curve conforms with that in figure 1: however, the winding number W is no longer the same physical quantity, being referred to a different axis.

Figure 2 shows such a case. Here two more instability bands have appeared on either side of the main band. These new bands have a maximum growth rate considerably smaller than that of the original elliptical instability and they span a much smaller range of angles θ. On further increases of $\omega_1(0)$ (with $|\boldsymbol{\omega}(0)| = 1$), many smaller instability bands appear and the three main bands continue to grow. Figures 3 to 5 show such cases. Some of these bands are not sufficiently large to show up clearly in figure 5, but figure 6 shows an enhanced resolution.

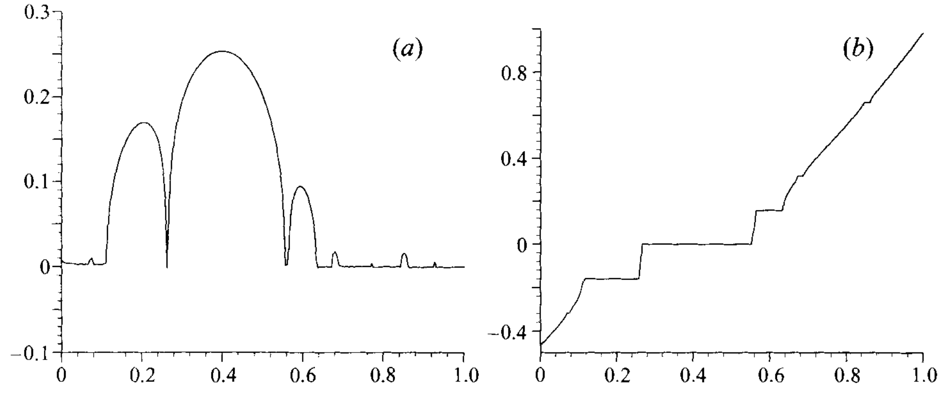

FIGURE 4. Same as figure 2 but for $\omega_1(0) = 0.968$ and $\omega_3(0) = 0.25$.

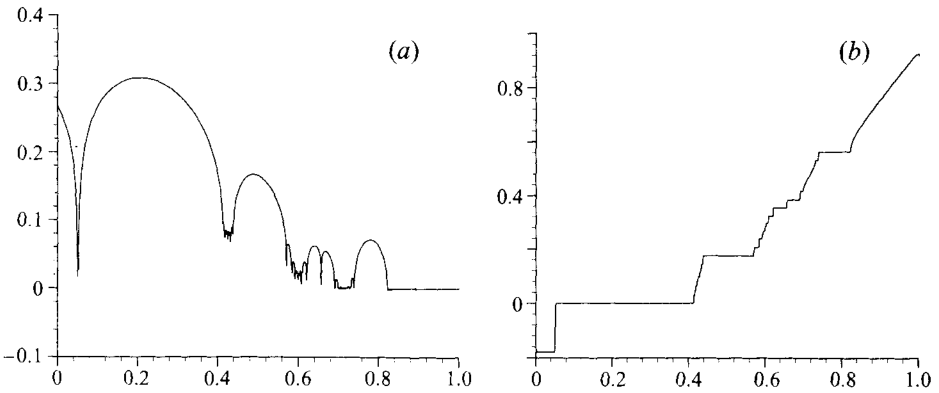

FIGURE 5. Same as figure 2 but for $\omega_1(0) = 0.866$ and $\omega_3(0) = 0.50$.

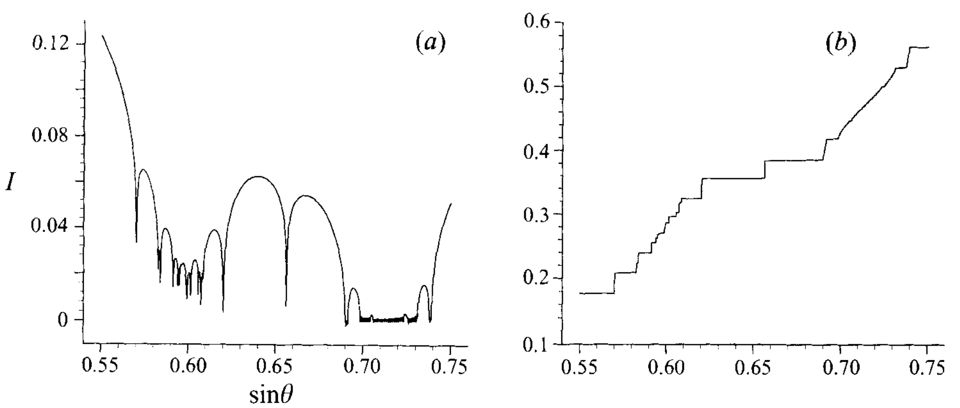

FIGURE 6. Sub-interval of figure 5, with improved resolution.

Such time-dependent flows are clearly highly unstable to a large set of disturbances. In figure 5, for example, instability is found for almost every angle between $\theta = 0$ and about $\sin^{-1} 0.81$. The diagrams shown in figure 6 have been produced using θ-intervals of 0.0005 over a region of figure 5 with $\sin\theta$ between 0.54 and 0.75. This figure shows clearly the complex structure in this interval, but even at this higher resolution it is impossible to show exactly the large number of stability bands that exist, for they will be everywhere dense in some intervals.

6. Discussion

The inherent instability of steady two-dimensional elliptical flows – unless stabilized by viscosity on small enough length scales – has implications for the persistence, or otherwise, of coherent eddies in turbulent flows. The more recent results of Craik & Allen (1992) and Bayly *et al.* (1996) on time-periodic flows, and their extension in the present paper to a new class of three-dimensional flows, show that the time-dependence of the basic flow can act to reinforce the elliptical instability through the creation of new parametric instability bands.

We expect very few swirling flows to be stable, unless the spatial dimensions are sufficiently small that viscous damping is considerable. Such flows as are inviscidly stable will either have circular streamlines or – as for elliptical flows in a rotating frame – will have vorticity close to zero in an inertial frame. Virtually any externally driven large-scale vortical flow within an asymmetric container is likely to be highly unstable to broad-band disturbances of the sort exemplified here. This implies that the natural state of such flows will be turbulent. Just one caveat should be mentioned: it is possible that weakly unstable, almost-circular flows can be stabilized by the application of periodic stretching and contraction with appropriate magnitude and frequency. This could come about through detuning of potentially unstable wavenumbers, when stretching and contraction alter their orientation sufficiently. Such stabilization could take place only at frequencies that do not admit other resonances directly due to the forcing. However, in the example just studied, where the oscillatory basic flow induces both tilting and stretching of disturbances, no such stabilization was observed and additional resonances were found.

The present work provides a new model for the local evolution and instability of more complex, possibly turbulent, flows. Also, flows within ellipsoidal configurations like those studied here are of considerable geophysical and astrophysical interest. For instance, Malkus' (1968) experiment led him to suggest that precession might sustain the Earth's magnetic field; and this provided an impetus for his later experiment (1989) (see also Vladimirov, Ribak & Tarasov 1993; Manasseh 1992) that clearly demonstrated elliptical instability. Lebovitz & Lifschitz (1996) have recently shown that the family of steady stellar structures known as self-gravitating Riemann ellipsoids are similarly subject to elliptical instability. There will be a corresponding family of time-dependent flows within an ellipsoidal configuration which, in view of the present study, may be expected to be even more unstable.

The universality of broad-band, inviscid, parametric instability within a wide class of flows with spatially uniform vorticity now seems firmly established. Other such flows are now amenable to similar analysis, employing the new method of Bayly *et al.* that is used here.

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